13/10/22 MATH4030 Tutorial

Reminders:

- Midterm next Tuesday 18th Oct. Midterm will cover everything up to and including Assignment 3.
- Assignment 3 due 11:59pm 21st Oct.

Outline for today's tutorial:

1) Gauss map and relation to Gaussian curvature.

2) Concept review for upcoming midterm.

- Concept review
  - regular curves, Frenet frame and Frenet formulas, Fundamental Thm of curves.
    - types of curves: cylindrical helix
  - regular surfaces:
    - definition, important examples (locally graphs, inverse images of regular values, ...)
    - first fundamental form, lengths and areas.
    - shape operator, second fundamental form Gauss curvature, mean curvature, principal curvatures

Gauss Map:

Last time, saw that an orientation of a regular surface M is a unit normal vector field N, that is,

- N changes smoothly with p
- $N \perp T_pM$ at each p
- N has unit length
- More concretely, if $X(u,v)$ is a parametrization $U \subset \mathbb{R}^2 \to M$, then N can be given by $\dfrac{X_u \times X_v}{|X_u \times X_v|}$.

Since N has unit length for each p, can actually view N as a map $M \to S^2$ the unit sphere, called the Gauss map.

$$p \mapsto N(p).$$

Actually, we can also identify with the map

$$N : U \subset \mathbb{R}^2 \to S^2 \text{ by } N(u,v) = N(X(u,v)).$$

By the definition of the shape operator S from last time, we have

$$S = -dN$$

The Gauss image is the image $A \subset S^2$ of M under the Gauss map.

e.g. What is the Gauss image of the plane?

What is the Gauss image of the circular cylinder?

The area of the Gauss image is related to the Gaussian curvature and is given by

$$\iint_M K \, dA.$$

This will show up later in the Gauss-Bonnet Theorem.

Concept review for upcoming midterm:

[I]: Regular Curves

Definition of a regular curve: $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ is a smooth curve in $\mathbb{R}^3$ if α is smooth. α is regular if $\alpha'(t) \neq 0$ for all $t \in I$.

Formula for arc-length of $l(\alpha)$ from t_0 to t_1 in I:

$$s(t) = \int_{t_0}^{t_1} |\alpha'(t)| \, dt$$

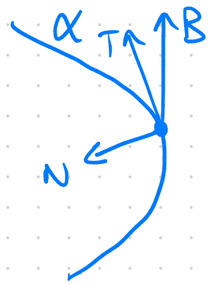
When we reparametrize α by arc-length s, $\alpha(s)$ has the property that $|\alpha'(s)| = 1$.

Frenet Frame: for α parametrized by arc-length s,

Tangent vector T is given by $T(s) = \alpha'(s)$

Curvature $K(s)$ is given by $K(s) = |T'(s)|$

Normal vector N is given by $N(s) = \dfrac{\alpha''(s)}{K(s)}$.



So $\alpha''(s) = K(s)N(s)$.

Binormal vector B is given by $B(s) = T(s) \times N(s)$

Torsion $\tau(s)$ is given by $B'(s) = -\tau(s)N(s)$. ← sign convention!

Then the Frenet formulas are given by

$T'(s) = K(s)N(s)$

$N'(s) = -K(s)T(s) + \tau(s)B(s)$ ← this one have to calculate.

$B'(s) = -\tau(s)R(s)$.

For straight lines, $\kappa(s) \equiv 0$

For plane curves, $\tau(s) \equiv 0$.

In general parameter t,

$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$$

$$\tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2}$$

Fundamental Thm of curves states that: Given $\kappa(s) > 0$, $\tau(s)$ smooth functions on (a,b), then there exists a regular curve $\alpha : (a,b) \to \mathbb{R}^3$ with $|\alpha'| = 1$ such that the curvature and torsion of α are given by κ, τ respectively. Moreover α is unique up to a rigid motion (translation and rotation).

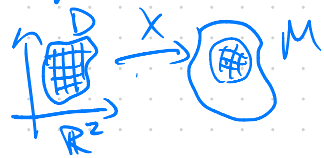
Examples of regular curves:

1) Cylindrical helix:

$$\alpha(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right), \quad c^2 = a^2 + b^2.$$

12: Regular Surfaces

Definition of regular surface: $M \subset \mathbb{R}^3$ is a regular surface if for any $p \in M$, there is an open nbhd U of p in M, an open set $D \subset \mathbb{R}^2$ and a parametrization $X: D \to U \cap M$ s.t.



1) X is smooth.

2) dX is full rank $\iff$ X_u, X_v are linearly independent

3) X is a homeomorphism from D to $M \cap U$

(i.e. X is bijective and both X, X^{-1} are continuous).

Examples of regular surfaces:

1) Sphere: $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$

Parametrization of northern hemisphere: $X(u,v) = (u, v, \sqrt{1-u^2-v^2})$.

Spherical coordinates $X(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

Stereographic projection...

2) Graphs of smooth functions: $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ smooth, D open, then $\text{graph} f := \{(x, y, f(x,y))\}$ is a regular surface.

3) Inverse image of regular values: $f: \mathbb{R}^3 \to \mathbb{R}$. $a \in \mathbb{R}$ is called a regular value of f if for $p \in \mathbb{R}^3$ s.t. $f(p) = a$, $\nabla f(p) \neq 0$.

Then the set $f^{-1}(a) := \{p \in \mathbb{R}^3 : f(p) = a\}$ is a regular surface.

4) Surfaces of revolution: $\alpha(u)$ be a regular curve in the y-z plane given by $\alpha(u) = (0, y(u), z(u))$, then the surface given by revolving the

curve around the z-axis is given by

$$X(u,v) = (y(u)\cos v,\ y(u)\sin v,\ z(u))$$

can do this for other axes as well.

4a) Torus: rotating circle $(y-a)^2+z^2=r^2$ about z-axis, obtain

$$X(u,v)=((r\cos u+a)\cos v,\ (r\cos u+a)\sin v,\ r\sin u)$$

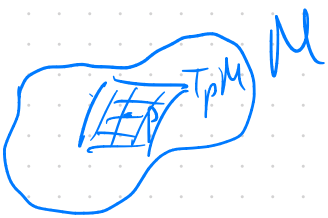
4b) Catenoid: rotate $y=\cosh x$

$$X(u,v) = (\cosh(u)\cos v,\ \cosh(u)\sin v,\ v)$$

. . . and more.

Tangent Space: For each $p \in M$, $p = X(u_0,v_0)$, the tangent space at p, $T_pM = \text{span}\{X_u(u_0,v_0),\ X_v(u_0,v_0)\}$.

Why is $\dim T_pM = 2$? Because X_u, X_v are linearly independent by the regularity condition.



Given X_u, X_v, the unit normal vector N normal to T_pM is given by

$$N = \frac{X_u \times X_v}{|X_u \times X_v|}.$$

Definition of the first fundamental form g: $g_p : T_pM \times T_pM \to \mathbb{R}$ by

$g_p(v,w) = \langle v, w \rangle$ where $\langle , \rangle$ is the standard inner product.

The coefficients of the first fundamental form are:

$E = \langle X_u, X_u \rangle$

$F = \langle X_u, X_v \rangle$

$G = \langle X_v, X_v \rangle$.

Can use first fundamental form to calculate:

1) length of a curve α from t_0 to t_1: $\alpha(t) = X(u(t), v(t))$

$$l(\alpha) = \int_{t_0}^{t_1} |\alpha'(t)|\,dt = \int_{t_0}^{t_1} \left(E(\alpha)\left(\frac{du}{dt}\right)^2 + 2F(\alpha)\frac{du}{dt}\frac{dv}{dt} + G(\alpha)\left(\frac{dv}{dt}\right)^2 \right)^{\frac{1}{2}} dt$$

2) area of a closed, bounded region R: $V = X^{-1}(R)$

$$A(R) = \iint_V |X_u \times X_v|\,du\,dv = \iint_V \sqrt{EG - F^2}\,du\,dv$$

Definition of the shape operator S_p wrt. N at p:

$v \in T_pM$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$, α regular curve, then

$$S_p(v) = -\frac{d}{dt} N(\alpha(t))\Big|_{t=0} = -dN_p.$$

As a map from $T_pM \to T_pM$, it is

- linear
- self-adjoint

$S_p(X_u) = -N_u$

$S_p(X_v) = -N_v$

Definition of the second fundamental form $\mathrm{II}: T_pM \times T_pM \to \mathbb{R}$

$$\mathrm{II}(v,w) = \langle S_p(v), w \rangle$$

It is a symmetric and bilinear form on T_pM.

The coefficients of II are

$$e = \mathrm{II}_p(X_u, X_u) = \langle S_p(X_u), X_u \rangle = -\langle N_u, X_u \rangle = \langle N, X_{uu} \rangle = \frac{\det(X_u, X_v, X_{uu})}{\sqrt{EG-F^2}}$$

$$f = \mathrm{II}_p(X_u, X_v) = \langle S_p(X_u), X_v \rangle = -\langle N_u, X_v \rangle = \langle N, X_{uv} \rangle = \frac{\det(X_u, X_v, X_{uv})}{\sqrt{EG-F^2}}$$

$$g = \mathrm{II}_p(X_v, X_v) = \langle S_p(X_v), X_v \rangle = -\langle N_v, X_v \rangle = \langle N, X_{vv} \rangle = \frac{\det(X_u, X_v, X_{vv})}{\sqrt{EG-F^2}}$$

Definition of Gauss Curvature and Mean Curvature:

$$K(p) = \text{determinant of } S_p \text{ matrix} = \frac{eg-f^2}{EG-F^2}$$

$$H(p) = \frac{1}{2}\text{trace of } S_p \text{ matrix} = \frac{1}{2}\,\frac{eG-2fF+gE}{EG-F^2}$$